Indian Statistical Institute, Bangalore Centre
B.Math. (I Year) : 2012-2013
Semester II : Semestral Examination
Probability Theory II

08.05.2013 Time: 3 hours. Maximum Marks : 100

Note: The paper carries 105 marks. Any score above 100 will be taken as 100. State clearly the results you are using in your answers.

1. (15 marks) Two friends agree to meet at a certain place between 12 noon and 1 p.m., with the first to reach agreeing to wait for the other person. Assume that their arrival times are independent, each having a uniform distribution over the assigned time interval. Let W denote the waiting time for the person reaching first. Find the probability density function of W.

2. (20 marks) Let U, W be independent random variables each having $N(1,1)$ distribution. Let $X = U - 2W, Y = U + 2W$. For any fixed $x \in \mathbb{R}$, find the conditional probability density function of Y given $X = x$. (Indicate clearly the family to which the conditional distribution belongs, mentioning explicitly all the relevant parameters.)

3. $(10 + 10 + 10 = 30$ marks) (i) Let $U_1, \cdots, U_n$ be i.i.d. random variables with common probability density function $f(\cdot)$. Derive the probability density function of the corresponding order statistics $(X_1, \cdots, X_n)$.

   (ii) In (i) above suppose $U_i, 1 \leq i \leq n$ have an exponential distribution with parameter λ. Define $Z_1 = X_1, Z_2 = X_2 - X_1, \cdots, Z_n = X_n - X_{n-1}$, where $X_i, 1 \leq i \leq n$ are as in (i). Find the probability density function of $(Z_1, \cdots, Z_n)$.

   (iii) With notation and assumptions as in (ii) above, find the marginal distributions of $Z_k, 1 \leq k \leq n$. Are they independent?

4. (20 marks) Let $\{p_n\}$ be a sequence of numbers in $(0,1)$ such that $\lim_{n\to\infty} np_n = \lambda$, where $\lambda > 0$. For $n = 1, 2, \cdots$ let X_n be a random variable having a binomial distribution with parameters n and p_n; let F_n denote the distribution function of X_n. Let X be a random variable having a Poisson distribution with parameter λ, with F denoting its distribution function. Using characteristic functions show that $\lim_{n\to\infty} F_n(z) = F(z)$ at every continuity point z of F.

5. (20 marks) Let $Y_1, Y_2, \cdots$ be a sequence of independent random variables each having an exponential distribution with parameter 1. Find

$$\lim_{n\to\infty} P\left(\frac{1}{2}n - \frac{1}{2\sqrt{3}}\sqrt{n} \le \sum_{i=1}^{n}[1-\exp(-Y_i)] \le \frac{1}{2}n + \frac{1}{2\sqrt{3}}\sqrt{n}\right).$$